Location of Modeling Errors Using Modal Test Data

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A unity check method is proposed to locate the physical positions of modeling errors in stiffness using modal test data. The method uses the cross-unity check between a flexibility matrix derived from modal test data and the analytical stiffness matrix to locate the errors; the method cannot determine the changes needed to correct the errors. The approach arrives at a formulation similar to one proposed previously by Ojalvo and Pilon. Effectiveness of the method is demonstrated through numerical examples using simulated mode test data and analytical stiffness matrices unreduced or reduced by Guyan reduction. Sensitivity of the method to lack of orthogonality in measured modes is discussed. Note that the flexibility matrix can also be derived from static test data instead of modal test data.

Introduction

DISCRETE linear dynamic models are widely used in structural design for predicting structural responses to dynamic loads. It is common practice in the aerospace industry to verify the model by a modal survey test to ensure accuracy in the prediction. Experience has shown that comparisons of analytically predicted modal data and those measured in the test almost invariably reveal deficiencies in the analytical model. The most common correctable modeling errors are erroneous assumptions, inadequate modeling detail, using wrong data, or simply typographical input errors. These errors are generally local in nature but very difficult to locate, especially for complex structures. State-of-the-art system identification techniques cannot identify these particular errors; the techniques only compensate for error effects by distributing model changes in some optimal fashion.[1–3] These changes often have little or no relation to the actual errors. An equivalent model results can reproduce the measured modes and frequencies but does not possess the physical parameters of the structure. The model, at best, may be suitable for evaluating the overall acceleration distribution but not for the detailed internal load distribution, essential for the evaluation of structural integrity. Berman[4] has provided a penetrating discusison on limitations of hte system identification.

In recent years, the question of how to locate errors in finite-element models using modal test data has been addressed by several researchers.[5–8] To address the same question, this paper will present a method which can locate the modeling errors in stiffness using the modal test data. Once the problem areas are located by the proposed method, the analyst can review the stiffness model in those areas and check pertinent input data to find the errors. The analyst may even resort to the system identification technique or some sensitivity analysis to determine the changes for some suspect parameters. The ultimate goal is to develop an analytical model which is both consistent with the test data and physically realistic. As for the mass matrix, the author believes that the dynamic behaviors of structures are generally less sensitive to mass modeling than to stiffness modeling and that mass modeling errors can be best corrected through mass property tests, which can accurately measure weights, moments of inertia, and centers of gravity.

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Before proceeding on the model correction, one needs to make sure that the measured modes are correct. Validity of the measured modes is typically judged by how well they can emulate the orthogonality characteristics of true normal modes.[9] Assuming that the test data have been satisfactorily verified, the proposed method checks the deviation of the product of a mode-test-derived flexibility matrix and the analytical stiffness matrix from unity. It is able to locate the errors because significant errors in the unity check occur only at degrees of freedom closely related to the modeling errors. The method arrives at a formulation which is a transposed form of one proposed by Ojalvo and Pilon[8] using a different approach. The approach adopted herein is based on the unity check principle, which lends itself to explanation of why the method works and makes it self-evident that the method can also be applied to using static test data to correct modeling errors.

Theoretical Formulation

It has been established that, with the mode shapes normalized to unit generalized mass, the stiffness and flexibility matricies are related to the modal data as follows[4,10]:

$$[k] = [m][\phi][\omega^2][\phi]^T[m] = [m]\left(\sum_{i=1}^{n} \omega_i^2 \{\phi_i\}\{\phi_i\}^T\right)[m] \tag{1}$$

$$[f] = [\phi][\omega^{-2}][\phi]^T = \sum_{i=1}^{n} \omega_i^{-2} \{\phi_i\}\{\phi_i\}^T \tag{2}$$

where

n = number of degrees of freedom
$[m]$ = mass matrix
$[k]$ = stiffness matrix
$[f]$ = flexibility matrix
$[\phi]$ = modal matrix
$\{\phi_i\}$ = ith modal vector
$[\omega^2]$ = diagonal modal frequency matrix
ω_i = ith modal frequency

The modal vectors and frequencies may come from analytical predictions or experimental measurements. From Eq. (1), it can be seen that modal contribution to the stiffness matrix increases as the frequency increases. In general, one has to measure all modes to obtain the stiffness experimentally. In practice, only the lowest few modes can be measured in modal survey tests for complex structures. On the other hand, Eq. (2) shows that the flexibility matrix converges rapidly as the modal frequency increases. Therefore, the flexibility matrix can be accurately obtained from the modal test. However, one cannot invert such a flexibility matrix to obtain the stiffness matrix because the flexibility matrix is singular when fewer than all modes are included.[10] Even if all modes are included, the matrix is still inherently ill conditioned making the inversion extremely sensitive to experimental errors.[11] Partition the modes in Eq. (2) as follows:

$$[f_a] = [\phi_{1a}\ \phi_{2a}]\begin{bmatrix}\omega_{1a}^{-2} & \\ & \omega_{2a}^{-2}\end{bmatrix}[\phi_{1a}\ \phi_{2a}]^T \tag{3}$$

where subscripts 1 and 2 represent the lower and higher modes, respectively, and a denotes analytically calculated. Replacing the lower modes and frequencies in Eq. (3) with their corresponding measured data, an experimentally derived flexibility matrix can be defined as follows:

$$[f_e] = [\phi_{1e}\ \phi_{2a}]\begin{bmatrix}\omega_{1e}^{-2} & \\ & \omega_{2a}^{-2}\end{bmatrix}[\phi_{1e}\ \phi_{2a}]^T \tag{4}$$

where subscript e denotes experimentally measured. In general, $[\phi_{1e}]$ is not orthogonal to $[\phi_{2a}]$, and $[f_e]$ may not even have the same rank as $[f_a]$. However, these should not pose problems unless one attempts to invert $[f_a]$.

Assuming the measured modes and frequencies are accurate, the analytical stiffness will also be accurate if the following unity condition is satisfied:

$$[f_e][k_a] = [I] \tag{5}$$

where $[I]$ is a unity matrix; otherwise, $[k_a]$ is in error. Since the stiffness matrix is banded or has strong diagonal terms, local modeling errors will significantly affect only stiffness coefficients for degrees of freedom directly connected to or closely related to the erroneously modeled areas. These affected coefficients will in turn influence only columns associated with the affected degrees of freedom in the product of $[f_e]$ and $[k_a]$. This localizing characteristic, illustrated in Fig. 1, can be used to locate modeling errors.

Physically, each column of $[k_a]$ represents a set of nodal forces due to a unit displacement at one degree of freedom while others are held fixed. Applying such nodal forces to the structure results in a nodal displacement pattern represented by the corresponding column in the $[f_e][k_a]$ product. Deviation of the displacements from the unity pattern indicates errors in the stiffness matrix. The deviation can be expressed by an error matrix $[E]$ defined as follows:

$$[E] = [f_e][k_a] - [I] \tag{6}$$

The error for each degree of freedom can be measured by the maximum absolute value of elements in the corresponding column of $[E]$, namely,

$$e_j = \max_i |e_{ij}| \tag{7}$$

where e_{ij} are elements of $[E]$. The e_j is a measurement of deflection errors caused by errors in the jth column of $[k_a]$; a large value of e_j indicates the jth degree of freedom is affected by modeling errors.

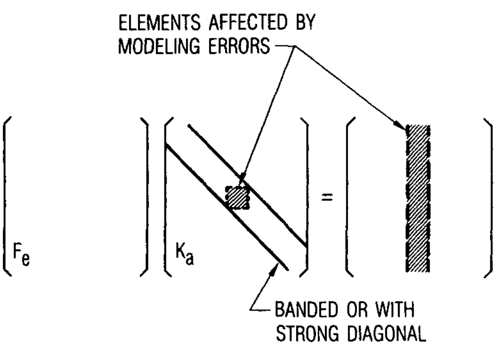

Fig. 1 Localizing characteristic of unity check errors.

Equation (6) can be cast into another form by first substituting $[I] = [f_a][k_a]$ into the equation:

$$[E] = ([f_e] - [f_a])[k_a] \tag{8}$$

Then, substitute Eqs. (4) and (3) into Eq. (8):

$$[E] = ([\phi_{1e}][\omega_{1e}^{-2}][\phi_{1e}]^T - [\phi_{1a}][\omega_{1a}^{-2}][\phi_{1a}]^T)[k_a] \tag{9}$$

Equation (9) is more economical to compute than Eq. (6) because it involves only the lower modes. Note that Eq. (9) is a transposed form of a method proposed by Ojalvo and Pilon.[8]

Numerical Examples

The proposed method will be tried out on numerical examples with known defects in the analytical models. The modal test data will be simulated by analytically predicted modes and frequencies using correct models with consistent mass matrices. Also, it is assumed that degrees of freedom measured in the test coincide with those of the analytical stiffness matrix. The examples include structures with stiffness matrix unreduced and with stiffness, along with the mass, reduced by the so-called Guyan reduction.[12] The objective is to see whether the unity check can indeed locate the modeling errors.

Truss with Unreduced Stiffness Matrix

The first example is a simply supported planar truss with 25 degrees of freedom, two translations at each of the free joints and one at the roller support (see Fig. 2). All degrees of freedom were retained in the analytical stiffness matrix and were measured in the modal test as well. Cross-sectional areas of members a and b were underestimated by 50% and 30%, respectively, in the analytical model. The unity check error e_j at each joint was defined as the greater of the two translational errors as defined by Eq. (7). The calculated errors are less than 0.01 at all joints except the four connected directly to members a and b. The significant errors are shown in Fig. 2 as coordinates perpendicular to the truss plane. It can be seen that the unity check can pinpoint exactly which members are in error. With only four modes measured, the calculated

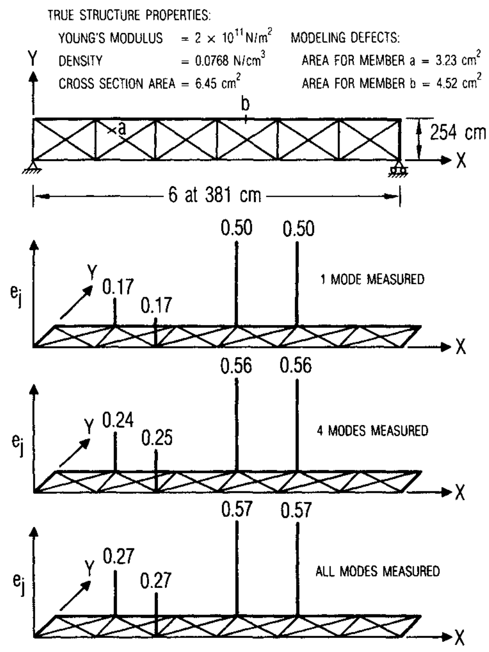

Fig. 2 Truss with unreduced stiffness matrix.

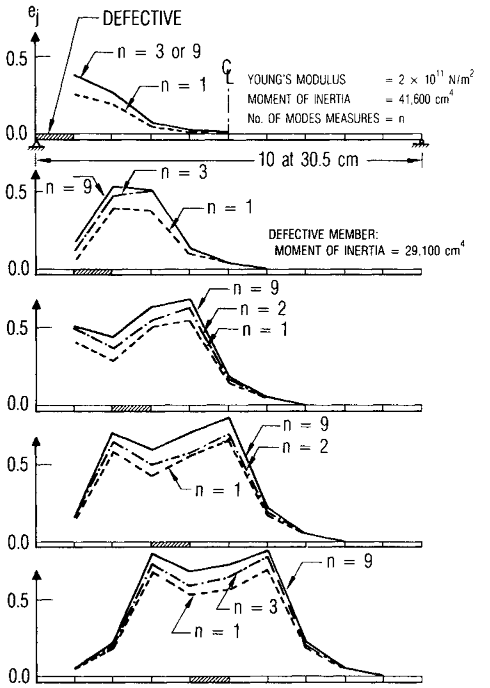

Fig. 3 Simply supported beam with reduced stiffness.

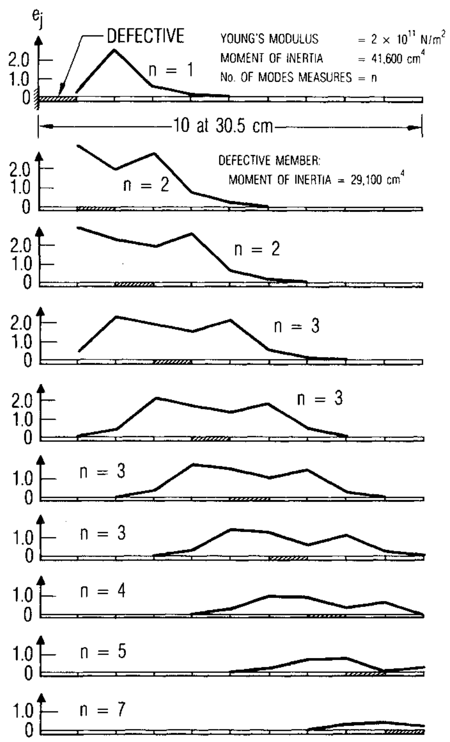

Fig. 4 Cantilever beam with reduced stiffness matrix.

errors already approach their exact values, values with all 25 modes measured. Even with only one mode measured, the errors already attain enough magnitudes to pinpoint the defectively modeled members indicating that the defective members have been sufficiently exercised by the first mode.

Simply Supported Beam

The second example is a uniform simply supported beam, perfectly rigid axially (see Fig. 3). The rotational degrees of freedom were eliminated by the Guyan reduction leaving only 9 transverse degrees of freedom in the stiffness matrix. A modeling error, a 30% underestimation of the moment of inertia, was deliberately introduced in one of the elements. The calculated unity check errors e_j are shown in Fig. 3 with the defective location varying from the left support to the center of span and with different numbers of modes measured. A striking revelation from the error distribution is that the errors no longer occur only at nodes adjoining the defective element; they overflow to neighboring nodes. This can be attributed to the redistribution effect of Guyan reduction. In an unreduced stiffness matrix, the coefficients err only at nodes adjoining the defective element. The reduction redistributes all coefficients, erroneous or not, associated with the eliminated degrees of freedom to the retained degrees of freedom which have direct load paths to the eliminated degrees of freedom; thus, the unity check errors spread out to other nodes. Fortunately, the redistributed forces are taken up mostly by the adjacent degrees of freedom; therefore, the redistributed errors stay mainly within nodes adjacent to or having significant direct load paths to the adjoining nodes.

It can also be seen that the unity check error converges rapidly with respect to the number of measured modes and that very few modes are needed to locate the modeling error. Note that errors at the adjoining nodes are sometimes lower than those of their adjacent nodes. This "notching" effect can be ascribed to the fact that the direct error from stiffness formulation and the redistributed error sometimes cancel each other. The effect at times becomes very prominent near the boundary.

Cantilever Beam

The preceding example was reanalyzed with cantilever boundary conditions (see Fig. 4). Ten transverse degrees of freedom were retained in the reduced stiffness matrix. The same modeling error was imposed in each element, one at a time, from the fixed end to the free end. Figure 4 illustrates that defects near the fixed end tend to show greater unity check errors than those near the free end. This can be attributed to the fact that a defect near the fixed end has a long moment arm to amplify the deflection error at the free end tip. The number of measured modes in each of the cases shown in Fig. 4 is selected such that the calculated errors have sufficiently converged to the exact values. It was found that many modes are required to sufficiently exercise defective elements near the free end. The notching effect, mentioned in the preceding example, also becomes very prominent when the defect occurs at the fixed end or one element away from the free end.

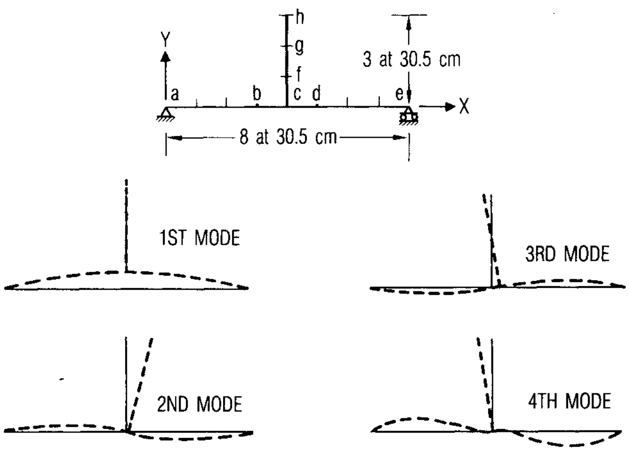

Fig. 5 Beam with a relatively stiff appendage.

Beam with a Relatively Stiff Appendage

The fourth example is a simply supported beam with a relatively stiff cantilever beam appended to the center of span (see Fig. 5). The material and cross-sectional properties are shown in Table 1. Rotational degrees of freedom were eliminated by the Guyan reduction with two translational degrees of freedom retained at each node. A 40% overestimate in element cf stiffness was introduced. The shapes of the first four correctly predicted modes are shown in Fig. 5. As in example 1, the unity check error is defined as the greater of the two translational errors. It can be seen from Fig. 6 that the errors overflow to adjacent nodes b, d, and g; however, the error at node g is much higher than those at nodes b and d. This can be attributed to the large difference between the beam stiffness and the appendage stiffness; the latter is 10 times as stiff as the former. The Guyan reduction redistributes the errors of eliminated degrees of freedom proportionally according to the relative stiffness of each element connected to the node. Therefore, very little of the error for rotational degree of freedom at node c is carried over to nodes b and d. It was also found that, with four measured modes, the errors approach the exact values.

Beam with a Relatively Flexible Appendage

The preceding example was reanalyzed with the appendage's stiffness reduced to 1/10 of the beam's. The same defect was induced in element cf. The first five correctly predicted modes are shown in Fig. 7. Unlike the preceding example, most of the error for rotational degree of freedom at node c is now redistributed by means of elements cb and cd resulting in more evenly distributed errors at nodes b, c, d, f, and g (see Fig. 8).

Orthogonality of Measured Modes

Unlike the simulated measured modes in the foregoing examples, real measured modes are never perfectly orthogonal. To address the question of how the lack of orthogonality affects the proposed unity check, the structure in example 5 will be re-examined using a set of more realistic "experimentally measured" modes. Let

$$[\phi_{1e}] = [\phi_1][C] \tag{10}$$

Table 1 Properties for a beam with a relatively stiff appendage

Young's modulus for beam	2×10^{11} N/m^2
Young's modulus for appendage	2×10^{12} N/m^2
Moment of inertia for beam and appendage	41,600 cm^4
Cross-sectional area for beam and appendage	353 cm^2
Mass per unit length for beam and appendage	277 N · s^2/m^2

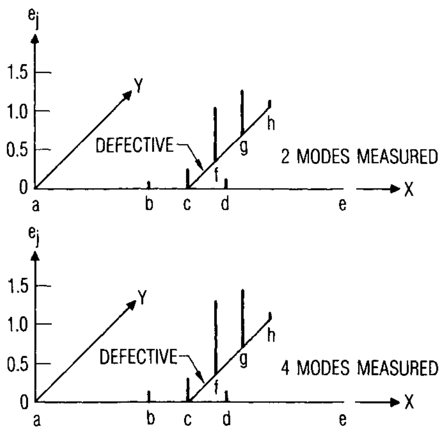

Fig. 6 Errors for beam with a relatively stiff appendage.

where

$[\phi_{1e}]$ = an incomplete set of measured modes
$[\phi_1]$ = the corresponding true modes
$[C]$ = corruption matrix

and the orthogonality matrix

$$[R] = [\phi_{1e}]^T[m][\phi_{1e}] \tag{11}$$

Assuming $[C]$ is symmetric, it can be proved[9] that

$$[C] = [R]^{1/2} \tag{12}$$

Using the correctly predicted modes as $[\phi_1]$ and a postulated $[R]$, more realistic measured modes can be synthesized by Eqs. (12) and (10).

It is widely accepted that, with the diagonal terms normalized to unity, off-diagonal terms of magnitude 0.10 or less in

Table 2 Orthogonality check for a beam with a relatively flexible appendage

Mode no.	1	2	3	4	5
1	1	0.02	−0.02	0.05	−0.02
2	0.02	1	−0.05	0.02	0.02
3	−0.02	−0.05	1	−0.02	0.10
4	0.05	0.02	−0.02	1	−0.02
5	−0.02	0.02	0.10	−0.02	1

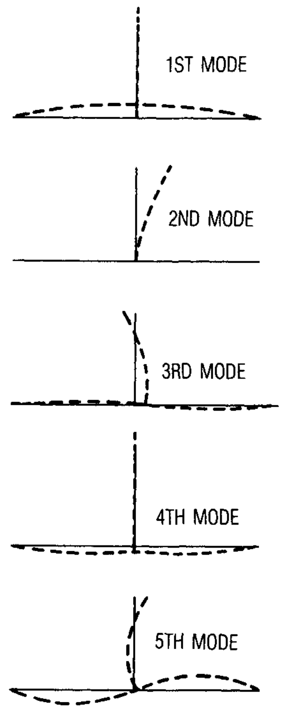

Fig. 7 Mode shapes for beam with a relatively flexible appendage.

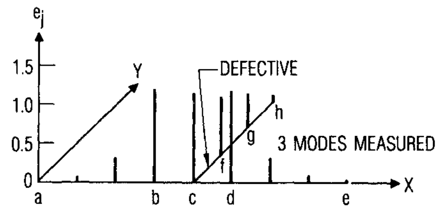

Fig. 8 Errors for beam with a relatively flexible appendage.

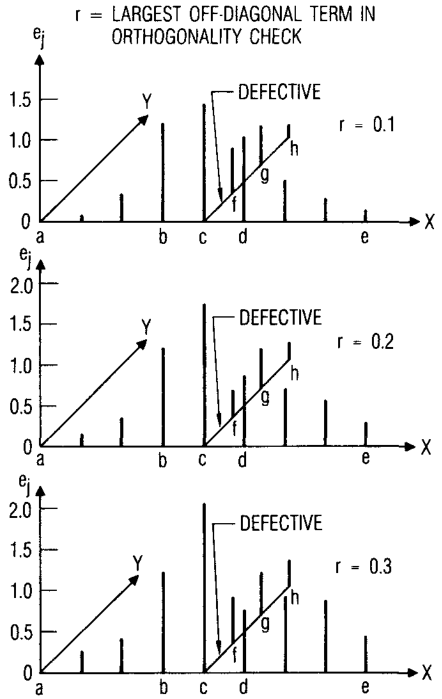

Fig. 9 Errors for beam with a relatively flexible appendage using unorthogonal measured modes.

[R] are indicative of adequate mode isolation. Therefore, an [R] matrix, shown in Table 2, with the largest off-diagonal term r equal to 0.10 is postulated. Two additional cases, $r = 0.20$ and 0.30, are generated by doubling and tripling the off-diagonal terms in Table 2. The recalculated unity check errors are shown in Fig. 9. The lack of orthogonality introduces additional unity check errors, which tend to obscure those caused by modeling defects. However, it appears that with $r = 0.10$, the unity check still maintains its ability to locate modeling errors.

Summary and Conclusions

The proposed method has been tested by the foregoing numerical examples. Important findings from these examples can be summarized as follows:

1) The unity check is a valid test for locating the physical stiffnesses that are in error.

2) A modeling error causes unity check errors in two ways: direct errors from the formulation of element stiffness and redistributed errors from the Guyan reduction. The direct errors appear only at nodes adjoining the defective element; large magnitudes of the redistributed errors show up only at nodes having direct load paths to the adjoining nodes.

3) The direct errors and redistributed errors sometimes cancel each other resulting in an apparent notch at the adjoining nodes.

4) The reduction redistributes the error of an eliminated degree of freedom proportionally according to the relative stiffness of each element connected to that degree of freedom.

5) The method will be most effective when the erroneous physical stiffnesses are well exercised by the measured modes.

6) Lack of orthogonality in measured modes introduces additional unity check errors. The common modal test criterion of requiring off-diagonal terms of the orthogonality matrix to be less than 0.10 appears to be adequate for the unity check method to locate modeling errors.

For simple structures, the flexibility matrix used in the unity check can be derived from static test data as well. Based on the examples tested, it appears that the method is effective, simple in concept, easy to implement, and thus suitable for practical application.

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